

The infrared fixed point of Landau gauge Yang-Mills theory: A renormalization group analysis

Axel Weber

Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,
Edificio C-3, Ciudad Universitaria, A. Postal 2-82, 58040 Morelia, Michoacán, Mexico

E-mail: axel@ifm.umich.mx

Abstract. The infrared behavior of gluon and ghost propagators in Landau gauge Yang-Mills theory has been at the center of an intense debate over the last decade. Different solutions of the Dyson-Schwinger equations show a different behavior of the propagators in the infrared: in the so-called scaling solutions both propagators follow a power law, while in the decoupling solutions the gluon propagator shows a massive behavior. The latest lattice results favor the decoupling solutions. In this contribution, after giving a brief overview of the present status of analytical and semi-analytical approaches to the infrared regime of Landau gauge Yang-Mills theory, we will show how Callan-Symanzik renormalization group equations in an epsilon expansion reproduce both types of solutions and single out the decoupling solutions as the infrared-stable ones for space-time dimensions greater than two, in agreement with the lattice calculations.

1. Introduction

The infrared regime of quantum chromodynamics is one of the prime examples of non-perturbative physics giving rise to such fascinating phenomena as the confinement of quarks and the dynamical breaking of chiral symmetry. The only method so far to obtain quantitatively reliable results directly from the underlying theory has been lattice gauge theory, and it has done so only in a limited fashion. For a long time, it was generally believed that analytical tools could not be of much help in the quantitative description of the aforementioned phenomena. This point of view is now slowly changing. In this contribution, before describing our own approach to the subject to which the title refers, we will try to give an overview, from our perspective, of what has been achieved so far in (the pure gauge sector of) infrared quantum chromodynamics with analytical and semi-analytical methods. Needless to say, progress with these methods has also been limited, but there have been some recent very promising developments.

The story begins in 1978 with the publication of Gribov's seminal work on the quantization of non-abelian gauge theories [1]. It was clear by that time that for the quantization of a gauge theory in the continuum the gauge needs to be fixed, and that, at least for perturbation theory, covariant gauges are particularly convenient. Specializing to the Landau gauge, the functional integral over the gauge field then includes two characteristic pieces: a functional delta distribution that implements the gauge condition $\partial_\mu A_\mu^a = 0$, and a functional determinant (Jacobian) that arises when one changes the integration variables from the general gauge fields A_μ^a to gauge equivalent fields \tilde{A}_μ^a that fulfill the gauge condition, and the corresponding gauge

transformations U , $A_\mu^a = (\bar{A}^U)_\mu^a$. The delta distribution can conveniently be written with the help of a bosonic auxiliary field called the Nakanishi-Lautrup field,

$$\delta(\partial_\mu A_\mu^a) = \int D[B] \exp \left(-i \int d^D x B^a \partial_\mu A_\mu^a \right), \quad (1)$$

while the Jacobian is represented in a local way as a functional integral over (fermionic scalar) ghost and antighost fields,

$$\det(-\partial_\mu D_\mu^{ab}) = \int D[c, \bar{c}] \exp \left(\int d^D x \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right). \quad (2)$$

We are generally working in D -dimensional Euclidean space-time, and the covariant derivative (in the adjoint representation) is defined as

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c. \quad (3)$$

In the functional integral of the theory (for gauge invariant operators) the integral over the gauge transformations U then factorizes and can consistently be omitted, and one is left in the pure gauge sector with an integral over A_μ^a , B^a , c^a and \bar{c}^a , and the (negative of the) Faddeev-Popov action

$$S_{\text{FP}} = \int d^D x \left(\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i B^a \partial_\mu A_\mu^a + \partial_\mu \bar{c}^a D_\mu^{ab} c^b \right) \quad (4)$$

in the exponent.

The original gauge symmetry of the Yang-Mills theory is manifestly broken by the gauge fixing procedure, however, the integration measure and the Faddeev-Popov action are invariant under the global fermionic BRST transformation s defined as

$$\begin{aligned} s A_\mu^a &= D_\mu^{ab} c^b, & s B^a &= 0, \\ s c^a &= \frac{1}{2} g f^{abc} c^b c^c, & s \bar{c}^a &= i B^a. \end{aligned} \quad (5)$$

An important property of this transformation (including the transformation of the matter fields) is its nilpotency, $s^2 = 0$. Note that the transformation of the gluon field has the form of an infinitesimal gauge transformation with the ghost field replacing the generator of the gauge transformation. The Ward identities corresponding to the BRST symmetry are the Slavnov-Taylor identities for the correlation functions.

As it turns out, the beta function for this theory is negative giving rise to the celebrated asymptotic freedom of non-abelian gauge theories and the applicability of perturbation theory in the ultraviolet regime. In the language of renormalization group theory, non-abelian gauge theories have a trivial fixed point (with vanishing renormalized coupling constant $g_R(\mu)$) which is ultraviolet-attractive. On the other hand, the renormalized coupling constant runs into a Landau pole at the characteristic scale Λ_{QCD} , and perturbation theory (around the trivial fixed point) becomes useless in the infrared, i.e., for energy-momentum scales of the order of and below Λ_{QCD} .

2. Gauge copies

In 1978, Gribov [1] observed that the gauge fixing procedure leading to the Faddeev-Popov action as described above is actually not consistent in non-abelian theories. The problem arises in the change of variables in the functional integral over the gauge field: the choice of a gauge-fixed field \bar{A}_μ^a , $\partial_\mu \bar{A}_\mu^a = 0$, that lies on the same gauge orbit as the initial field A_μ^a , $A_\mu^a = (\bar{A}^U)_\mu^a$

for an appropriate gauge transformation U , is not unique. In other words, there exist different gauge fields, called gauge copies or Gribov copies, that fulfill the gauge condition and are, nevertheless, connected by a gauge transformation. In geometric language, the gauge orbits intersect the hypersurface defined by $\partial_\mu \bar{A}_\mu^a = 0$ more than once.

In particular, if we look at an infinitesimal gauge transformation generated by ω^a , we have

$$\partial_\mu (A^\omega)_\mu^a = \partial_\mu A_\mu^a + \partial_\mu D_\mu^{ab} \omega^b. \quad (6)$$

Then it is possible to have both, A_μ^a and $(A^\omega)_\mu^a$, fulfill the gauge condition if ω^a is a zero mode of the Faddeev-Popov operator $(-\partial_\mu D_\mu^{ab})$. Hence Gribov's proposal to restrict the integration over the gauge field to the region where the Faddeev-Popov operator is positive definite, now called the (first) Gribov region and denoted by Ω . Note that at the boundary of the Gribov region, the (first) Gribov horizon $\partial\Omega$, we have that $\det(-\partial_\mu D_\mu^{ab}) = 0$ because the Faddeev-Popov operator develops a zero mode there.

Although gauge copies which are related by an infinitesimal gauge transformation are excluded by the restriction to Ω , it is not clear that finitely related gauge copies cannot exist within Ω . In fact, it is now known that such copies do exist in Ω , and the consequent thing to do would be to restrict the integration over the gauge fields further to a smaller region that is called the fundamental modular region Λ . However, the restriction to Λ is very hard to implement, in numerical calculations (lattice gauge theory with Landau gauge fixing) and even more so in analytical (or semi-analytical) calculations. Fortunately, according to an argument by Zwanziger [2], the difference between the restrictions to Ω or to Λ has no effect on the correlation functions.

For the rest of this contribution, we will hence consider the restriction of the integral over the gauge field to the Gribov region Ω . One of the important implications of this restriction as compared to an unrestricted integral over the gauge field, according to an argument by Gribov [1], is the impossibility of a Landau pole for the renormalized coupling constant to occur (at a momentum scale different from zero). Realizing the restriction to Ω in an approximate way, Gribov was furthermore able to predict the infrared behavior of the gluon and ghost propagators. Here we will present an exact way to implement the restriction of the integral over the gauge fields which is due to Zwanziger [3] and based on Gribov's ideas, and come back to the propagators later.

Zwanziger found a way to realize the (Heavyside) theta functional for the restriction of the integral over the gluon field to the Gribov region as a Gaussian (in the thermodynamic limit):

$$\int_\Omega D[A] = \int D[A] \theta(-\partial_\mu D_\mu^{ab}) = \int D[A] \exp \left(-\gamma^4 \int d^D x h(x) \right), \quad (7)$$

with the horizon function $h(x)$ defined as

$$h(x) = g^2 f^{abc} f^{cde} A_\nu^b [(-\partial_\mu D_\mu)^{-1}]^{ad} A_\nu^e. \quad (8)$$

For the correct normalization, the horizon condition

$$\langle h(x) \rangle = 4(N^2 - 1) \quad (9)$$

has to be fulfilled (in an $SU(N)$ gauge theory). This condition fixes the new parameter γ with the dimension of mass (in terms of Λ_{QCD} , or vice versa).

Now $h(x)$ is a complicated nonlocal functional because it contains the inverse of the Faddeev-Popov operator. A local form of the theory, necessary for the application of the usual quantum field theoretical machinery, can be achieved by introducing two additional auxiliary vector fields (and the corresponding antifields), one of them bosonic ϕ_μ^{ab} and the other fermionic ω_μ^{ab} , each

carrying two color indices (in the adjoint representation). The gauge sector of the theory including the restriction to the first Gribov region is then described by a functional integral over A_μ^a , B^a , c^a , \bar{c}^a , ϕ_μ^{ab} , $\bar{\phi}_\mu^{ab}$, ω_μ^{ab} and $\bar{\omega}_\mu^{ab}$, and the exponential of the (negative of the) Gribov-Zwanziger action [4]

$$S_{\text{GZ}} = S_{\text{FP}} - \int d^D x \left(\partial_\mu \bar{\phi}_\nu^{ac} D_\mu^{ab} \phi_\nu^{bc} - \partial_\mu \bar{\omega}_\nu^{ac} D_\mu^{ab} \omega_\nu^{bc} + g f^{abc} \partial_\mu \bar{\omega}_\nu^{ad} D_\mu^{be} c^e \phi_\nu^{cd} \right) - \gamma^2 \int d^D x \left(g f^{abc} A_\mu^a \phi_\mu^{bc} + g f^{abc} A_\mu^a \bar{\phi}_\mu^{bc} + 4(N^2 - 1)\gamma^2 \right), \quad (10)$$

see eq. (4). The horizon condition can be rewritten in terms of the auxiliary fields as

$$\langle g f^{abc} A_\mu^a (\phi_\mu^{bc} + \bar{\phi}_\mu^{bc}) \rangle = 8(N^2 - 1)\gamma^2. \quad (11)$$

The Gribov-Zwanziger action leads to the tree-level gluon propagator (in momentum space)

$$\langle A_\mu^a(p) A_\nu^b(-q) \rangle = \frac{p^2}{p^4 + \lambda^4} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p - q), \quad (12)$$

with the characteristic scale λ defined as

$$\lambda^4 = 2g^2 N \gamma^4. \quad (13)$$

For $p^2 \gg \lambda^2$, one recovers the usual (tree-level) ultraviolet behavior, while the propagator is heavily suppressed for $p^2 \ll \lambda^2$ and even tends to zero in the limit $p^2 \rightarrow 0$. The tree-level ghost propagator, on the other hand, behaves like

$$\langle c^a(p) \bar{c}^b(-q) \rangle \propto \frac{1}{p^4} \delta^{ab} (2\pi)^D \delta(p - q) \quad (14)$$

in the infrared regime $p^2 \ll \lambda^2$. The divergence in the limit $p^2 \rightarrow 0$ is much stronger than for the tree-level ghost propagator in Faddeev-Popov theory that does not take the restriction to the Gribov region into account. It is clear, then, that the effect of the restriction to the Gribov region is rather drastic for the infrared behavior of the propagators. The forms (12) and (14) of the propagators have already been found by Gribov in his approximate description [1].

3. Dyson-Schwinger equations

In an initially independent development, the Dyson-Schwinger equations of quantum chromodynamics were approximately solved in the infrared regime. As in the previous section, we here focus on the pure gauge sector, i.e., Yang-Mills theory. The Dyson-Schwinger equations are derived from the Faddeev-Popov action (4), and the restriction to the Gribov region is not implemented in any way. While this is clear from the independence of this development, the reader is justified to wonder why it would make any sense to present the solution of Dyson-Schwinger equations that ignore the restriction to the Gribov region, when we have argued at the end of the last section that the restriction to the Gribov region has a severe effect on the infrared behavior of the propagators.

However, as Zwanziger has clarified [5], the restriction to the Gribov region actually leaves the Dyson-Schwinger equations unchanged: Dyson-Schwinger equations are derived from the fact that the functional integral over a (functional) derivative vanishes. When one restricts the functional integral over the gauge field in the Faddeev-Popov formulation to the Gribov region Ω , contributions from the boundary $\partial\Omega$ are to be expected. However, there are no such

$$\begin{aligned} \left(\text{---}\bullet\text{---} \right)_p^{-1} &= Z_A(p^2 \delta_{\mu\nu} - p_\mu p_\nu) - \text{---}\bullet\text{---} \text{---}\bullet\text{---} \text{---}\bullet\text{---} \\ \left(\text{---}\bullet\text{---} \right)_p^{-1} &= Z_c p^2 - \text{---}\bullet\text{---} \text{---}\bullet\text{---} \text{---}\bullet\text{---} \end{aligned}$$

Figure 1. Simplified Dyson-Schwinger equations for the gluon and ghost propagators containing the dominant contributions in the infrared regime.

contributions in this case since the integrand vanishes on $\partial\Omega$ due to the presence of the Faddeev-Popov determinant as remarked earlier, and the resulting Dyson-Schwinger equations are the same as for the unrestricted functional integral.

Solutions to the Dyson-Schwinger equations for the gluon and ghost propagators have been found, in a certain approximation, first in refs. [6, 7]. The results are an infrared suppressed gluon propagator and an infrared enhanced ghost propagator as compared to the tree-level propagators of Faddeev-Popov theory, which is qualitatively consistent with the results (12) and (14) of the Gribov-Zwanziger framework. Then, with hindsight, we can simplify the Dyson-Schwinger equations in the infrared regime by omitting the diagrams that contain gluon loops, or in general, in the Dyson-Schwinger equation for a given correlation function only keep the diagrams with the biggest number of ghost loops since the latter dominate the infrared behavior. The dominance of the diagrams with the biggest number of internal ghost propagators in the infrared regime is sometimes referred to as ghost dominance.

Furthermore, Taylor's non-renormalization theorem [8] allows us to replace the full ghost-gluon vertex by the bare one. That this replacement is valid to a good approximation even in the non-perturbative regime has been checked in lattice calculations [9]. With these simplifications, the Dyson-Schwinger equations for the gluon and ghost propagators take the form depicted in fig. 1. This simplified system of equations not only correctly describes the extreme infrared limit of the theory, but it also gives a qualitative description of the extreme ultraviolet, asymptotically free regime, and it can thus serve as a useful laboratory for the study of several properties of the complete theory [10]. For a quantitatively correct description of the ultraviolet regime, in particular the correct value of the (one-loop) beta function, however, one needs to take additional contributions into account in the first equation of fig. 1, and more sophisticated approximation schemes have been devised to this end [11, 12].

If one is exclusively interested in the extreme infrared regime, the simplified Dyson-Schwinger equations can even be solved analytically. In order to describe the solutions, we will introduce some notation: we write the propagators in terms of their dressing functions $G(p^2)$ and $F(p^2)$,

$$\begin{aligned}\langle A_\mu^a(p) A_\nu^b(-q) \rangle &= \frac{G(p^2)}{p^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p - q), \\ \langle c^a(p) \bar{c}^b(-q) \rangle &= \frac{F(p^2)}{p^2} \delta^{ab} (2\pi)^D \delta(p - q).\end{aligned}\tag{15}$$

For the analytical solutions, we make power-like ansaetze for the dressing functions (this behavior

is familiar from the theory of critical phenomena):

$$G(p^2) \propto (p^2)^{-\alpha_G}, \quad F(p^2) \propto (p^2)^{-\alpha_F}. \quad (16)$$

Inserting the ansatz (16) in the system of approximate Dyson-Schwinger equations in fig. 1, one obtains, for any dimension D between two and four, *two* consistent solutions known as scaling solutions [5, 13]. One of these solutions is

$$\alpha_F(D) = \frac{D-2}{2}, \quad \alpha_G(D) = -\frac{D}{2} \quad (17)$$

in dimension D , for $2 < D < 4$ (this solution is not entirely consistent in dimensions two and four due to logarithmic corrections). The infrared exponents of the other scaling solution have to be obtained, for general D , from the solution of a transcendental equation. A good approximation (within 2% error for any D with $2 \leq D \leq 4$) is given, however, by the simple formulae

$$\alpha_F(D) = \frac{D-1}{5}, \quad \alpha_G(D) = -\frac{16-D}{10}. \quad (18)$$

For both solutions (also the exact version of the second solution), the infrared exponents fulfill the sum rule

$$\alpha_G + 2\alpha_F = \frac{D-4}{2} \quad (19)$$

in any dimension D . The solutions also have in common that $\alpha_F > 0$ and $1 + \alpha_G < 0$ (for $D > 2$ in the first solution), implying that the ghost propagator diverges more strongly in the infrared than the (Faddeev-Popov) tree-level propagator, while the gluon propagator is strongly suppressed in the infrared, i.e., it tends to zero in the limit $p^2 \rightarrow 0$, just like the tree-level propagators in the Gribov-Zwanziger framework.

Taking into account that the Dyson-Schwinger equations are not altered by the restriction of the functional integral to the Gribov region as argued at the beginning of this section, we interpret the solutions (17) and (18) as (approximately) incorporating the quantum corrections to the tree-level results (12) and (14) presented in the previous section. It is also clear that the power-like behavior (16) of the solutions can only be obtained by summing up the quantum corrections (in some approximation) to *any* loop order. On the other hand, it is a priori not so clear which of the two solutions (17) and (18) is the physically relevant one for general dimension D . Also, one would like to have a systematic scheme to improve on the approximations employed in the derivation of the Dyson-Schwinger equations in fig. 1, and it has turned out to be very difficult to formulate such a scheme.

An important consequence of Taylor's non-renormalization theorem is the natural definition of a running coupling constant as

$$g_R^2(p^2) = (p^2)^{(D-4)/2} G(p^2) F^2(p^2) g^2. \quad (20)$$

The factor $(p^2)^{(D-4)/2}$ compensates for the dimension of the bare coupling constant g^2 (the dressing functions $G(p^2)$ and $F(p^2)$ are by definition dimensionless), so that $g_R^2(p^2)$ is a dimensionless quantity. Then it is clear from the sum rule (19) that the running coupling constant tends to a finite fixed point value in the infrared limit. This value can be determined analytically for the two scaling solutions (17) and (18), and it turns out to vary continuously with the dimension D . As an example, we give the (exact) fixed point value in $D = 4$ for the second solution (18),

$$\frac{Ng_R^2}{4\pi} = 8.915 \dots, \quad (21)$$

and mention that g_R^2 tends to zero in the limit $D \rightarrow 2$ for the first solution (17).

Finally, we mention that the same results have also been obtained (in $D = 4$ dimensions) with a different technique, the functional renormalization group equations [12, 14] which, however, have many features in common with the Dyson-Schwinger equations.

4. Decoupling solutions

One of the characteristic features of the scaling solutions (including their tree-level forms (12), (14)) is the infrared enhancement of the ghost propagator. In fact, it may be shown [4] that the horizon condition (9) is equivalent to

$$\lim_{p^2 \rightarrow 0} F^{-1}(p^2) = 0, \quad (22)$$

the infrared divergence of the ghost dressing function. It is well understood by now that implementing the condition (22) necessarily leads to the scaling solutions, be it in the tree-level form (12) and (14) or in the “resummed” form (17) or (18).

There is another important aspect of the Gribov-Zwanziger framework [15]: the horizon condition is identical to the Kugo-Ojima confinement criterion [16, 17] which guarantees (in Faddeev-Popov theory) the existence of a global color charge that is BRST-exact and hence implies confinement in the sense that all physical states are color singlets. Kugo and Ojima construct the physical Hilbert space of non-abelian gauge theories starting from BRST invariance and show the unitarity of the S -matrix [16]. The equivalence of the confinement criterion (and hence the horizon condition) to the infrared divergence (22) of the ghost dressing function in the Landau gauge has been shown in this framework, too [17].

At first sight, everything seems to fit together nicely: the restriction of the functional integral over the gluon field to the Gribov region, necessary because of the existence of gauge copies in non-abelian gauge theories, leads to the horizon condition or the Kugo-Ojima criterion and hence to the confinement of color charges. However, there is a fundamental flaw in this argument: the BRST symmetry of the Faddeev-Popov action can be extended to the Gribov-Zwanziger action (10) only for $\gamma^2 = 0$, in which case the integration over the gluon field is effectively unrestricted, see eq. (7). In other words, the restriction to the Gribov region breaks the BRST invariance on which the whole Kugo-Ojima construction rests.

There is a simple intuitive argument that shows how the breakdown of BRST invariance is an inevitable consequence of the restriction to the Gribov region [18]: the BRST transformation of the gluon field has the form of an infinitesimal gauge transformation, as already remarked following eq. (5). But close to the Gribov horizon, an infinitesimal gauge transformation can relate gauge copies, see eq. (6), and hence the transformed field falls outside the Gribov region.

The violation of BRST symmetry is not in itself a catastrophe for the theory. Many of us have grown up thinking of gauge invariance and BRST invariance as synonyms, mostly because the (global) BRST symmetry implies the Slavnov-Taylor identities. However, this is really only true for the Faddeev-Popov action and an unrestricted functional integral over the gluon field. If we restrict the integration to the Gribov region, as it turns out to be necessary in order to avoid gauge copies, the (gauge fixed) quantum theory is not invariant under BRST transformations any more. Of course, this is a serious problem for the construction of the Hilbert space of the theory, just because the standard construction (of Kugo and Ojima) is based on BRST symmetry. A solution to this problem has not yet been devised.

But there is yet another twist to our story. With the analytical and semi-analytical predictions of the propagators in the infrared regime at hand, it seemed natural to corroborate these predictions with the help of completely non-perturbative lattice calculations. While it is not necessary (nor natural) to fix the gauge in these calculations, it is also not too difficult to implement the Landau gauge on the lattice: for a given gauge field configuration A_μ^a , consider the functional

$$\int d^D x (A^U)_\mu^a (A^U)_\mu^a \quad (23)$$

for all possible gauge transformations U , i.e., along the gauge orbit that contains A_μ^a . Then the extrema (or saddle points) of this functional along the orbit correspond to the fields $\bar{A}_\mu^a = (A^U)_\mu^a$

that fulfill the gauge condition $\partial_\mu \bar{A}_\mu^a = 0$, the local minima correspond to fields \bar{A}_μ^a in the first Gribov region Ω , and the absolute minima to fields in the fundamental modular region Λ . For the huge lattices that are necessary to measure the infrared behavior of the propagators, it is extremely difficult (numerically costly) to determine the absolute minimum of the functional (23) along a given gauge orbit, however, it is relatively easy to find a local minimum and hence to restrict the gauge fields to the Gribov region.

The first lattice results for the propagators in the Landau gauge (with the restriction to the Gribov region Ω) show a clear increase of the ghost dressing function $F(p^2)$ towards the infrared, and a maximum and subsequent marked decrease of the gluon dressing function $G(p^2)$ as one follows p^2 from the ultraviolet to the infrared, in agreement with the scaling solutions of the Dyson-Schwinger equations. However, it was difficult to discern for a long time whether the ghost dressing function would actually diverge in the limit $p^2 \rightarrow 0$ and whether the gluon propagator itself would vanish in the same limit.

Meanwhile, it was discovered that the Dyson-Schwinger equations of the theory permit another type of solutions now known as decoupling solutions [19, 20, 21] (see also [12]). In these solutions, the ghost propagator is infrared enhanced only by a finite constant, i.e., the ghost dressing functions tends toward a finite constant in the limit $p^2 \rightarrow 0$, while the gluon propagator has a finite (nonzero) limiting value similarly to the propagator of a massive particle. Then one still has ghost dominance in the sense that the gluon propagator is infrared suppressed in comparison with the ghost propagator, and the reduction of the full Dyson-Schwinger equations to the approximation represented in fig. 1 is justified in the infrared regime.

In terms of the infrared exponents defined in eq. (16), one has for the decoupling solutions that

$$\alpha_F = 0, \quad \alpha_G = -1 \quad (24)$$

in any dimension D (coincident with the first scaling solution (17) in the limit $D \rightarrow 2$). In particular, the decoupling solutions do not respect the sum rule (19) for the exponents (except at $D = 2$). Furthermore, the running coupling constant defined in eq. (20) tends to zero in the limit $p^2 \rightarrow 0$ for the decoupling solutions (for dimensions $D > 2$).

The discovery of the decoupling solutions of the Dyson-Schwinger equations initiated a long and often passionate debate in the community about the true infrared behavior of the Landau gauge propagators. In 2007, at the lattice conference in Regensburg, three lattice groups presented their results on the infrared behavior of the propagators in Landau gauge Yang-Mills theory on very large lattices [22, 23, 24]. The majority of the community has interpreted these results as confirming the *decoupling solutions* of the Dyson-Schwinger equations in three and four space-time dimensions. In the same year, lattice calculations in *two* dimensions [25] confirmed the realization of the second scaling solution (18).

At first sight, the decoupling solutions are at odds with the tree-level propagators (12) and (14) and the horizon condition in the form (22) that result in the Gribov-Zwanziger scenario. However, indications were found in ref. [18] (see also [26]) that the auxiliary fields form a condensate $\langle \bar{\phi}_\mu^{ab} \phi_\mu^{ab} \rangle$ (among other possible condensates). As a consequence, the horizon condition (11) does *not* imply the infrared divergence (22) of the ghost dressing function any more, and through the linear coupling of the fields ϕ_μ^{ab} and $\bar{\phi}_\mu^{ab}$ to the gluon field, see eq. (10), the condensate leads to a mass term for the gluons.

In the simplest case [18] in this so-called refined Gribov-Zwanziger framework, i.e., with the condensates taken into account, the tree-level propagators are of the form

$$\langle A_\mu^a(p) A_\nu^b(-q) \rangle = \frac{p^2 + M^2}{p^4 + M^2 p^2 + \lambda^4} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p - q) \quad (25)$$

(see eq. (13) for the definition of the characteristic scale λ), and

$$\langle c^a(p)\bar{c}^b(-q)\rangle \propto \frac{1}{p^2} \delta^{ab} (2\pi)^D \delta(p-q) \quad (26)$$

in the infrared regime. The gluon propagator can also be written in terms of an effective momentum-dependent mass $M_{\text{eff}}(p^2)$,

$$\frac{p^2 + M^2}{p^4 + M^2 p^2 + \lambda^4} = \frac{1}{p^2 + M_{\text{eff}}(p^2)}, \quad (27)$$

where

$$M_{\text{eff}}(p^2) = \frac{\lambda^4}{p^2 + M^2}. \quad (28)$$

In four space-time dimensions, a successful fit of the tree-level gluon propagator in the refined Gribov-Zwanziger framework with an additional $\langle A_\mu^a A_\mu^a \rangle$ -condensate,

$$\langle A_\mu^a(p) A_\nu^b(-q) \rangle = \frac{p^2 + M^2}{p^4 + (M^2 + m^2)p^2 + \lambda^4 + M^2 m^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p-q), \quad (29)$$

to the lattice data in the infrared and intermediate momentum regimes has been performed in ref. [27] (see also [28]). It is important to clarify that the different mass parameters that have appeared so far are not additional parameters of the theory, but can rather be determined, at least in principle, as multiples of Λ_{QCD} from a calculation of the effective potential [18].

At this point, from the perspective of the Dyson-Schwinger equations, the decoupling solution as well as both of the scaling solutions are consistent and valid solutions, and from these equations alone there is no reason to prefer one of the solutions over the others. In the refined Gribov-Zwanziger framework, on the other hand, the decoupling solution with the tree-level propagators (25) or (29) and (26) is preferred according to the lowest-order calculation of the effective potential for various local composite operators [26]. The problem with this latter approach is rather of a technical nature: it appears extremely cumbersome to carry the calculations to higher orders.

We finally mention an important property of the scaling as well as the decoupling solutions: all the solutions found so far violate positivity, i.e., the spectral function in the Källén-Lehmann representation of the gluon propagator is not positive semidefinite implying that the gluon *cannot* be a physical (massive or massless) particle (see, e.g., refs. [7, 12, 18, 29, 30]).

5. Callan-Symanzik equations

5.1. A gluon mass term

For the rest of this contribution, we will describe a recent approach to the subject from a renormalization group perspective [31]. We start from the Dyson-Schwinger equations of Faddeev-Popov theory which, as explained before, do not change when one restricts the functional integral over the gluon field to the Gribov region Ω . Since the iterative solution of the Dyson-Schwinger equations generates the perturbative expansion, the same is true for the perturbative series. On the other hand, the (re)normalization conditions that define the quantum theory can certainly change as a result of the restriction.

One of the most important consequences of the restriction to the Gribov region, as mentioned in the last section, is the breaking of BRST invariance. The resulting violation of the Slavnov-Taylor identities leads, almost inevitably, to the generation of a gluon mass term as the only possible properly relevant term in the (effective) action with the field content of Faddeev-Popov theory, i.e., without the additional auxiliary fields of the Gribov-Zwanziger action. Note that

a ghost mass term cannot be induced for the same kinematical reasons that lead to Taylor's non-renormalization theorem. We will, hence, include a gluon mass term

$$\int d^D x \frac{1}{2} A_\mu^a m^2 A_\mu^a \quad (30)$$

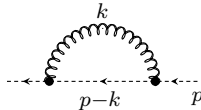
in the Faddeev-Popov action.

Such an action has been considered first, as a special case among a much more general class of actions, in ref. [32] where the extension of the BRST transformation and the Slavnov-Taylor identities to the massive case is discussed (however, the Nakanishi-Lautrup field is not introduced in ref. [32]). The Faddeev-Popov action with a gluon mass term has also been used in a recent perturbative calculation [33, 34] to which we come back later. Since the mass term originates from the breaking of the BRST symmetry as a consequence of the restriction to the Gribov region, we expect the (renormalized) mass parameter to be of the order of the characteristic scale Λ_{QCD} . It is then clear that the introduction of the mass term has no important effect in the ultraviolet regime, i.e., for momenta $p^2 \gg \Lambda_{\text{QCD}}^2$. In particular, the value of the ultraviolet beta function is unchanged and, of course, the theory is asymptotically free. This is consistent with the fact that the restriction to the Gribov region is irrelevant for the perturbative expansion in the ultraviolet regime where only “small” fields A_μ^a contribute.

Now, we would at least naively expect that in the extreme infrared regime $p^2 \ll m^2$ the mass term (30) dominates over the other term in the action that is quadratic in A_μ^a ,

$$\int \frac{d^D p}{(2\pi)^D} A_\mu^a(-p) (p^2 \delta_{\mu\nu} - p_\mu p_\nu) A_\nu^a(p). \quad (31)$$

In fact, this is true also for the loop corrections as we will now illustrate with the help of a relevant example: consider the one-loop ghost self-energy



$$= N g^2 \delta^{ab} \int \frac{d^D k}{(2\pi)^D} p_\mu \frac{1}{(p-k)^2} (p_\nu - k_\nu) \frac{1}{k^2 + m^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (32)$$

It is worth mentioning that the tree-level gluon propagator in Landau gauge is exactly transverse in momentum space even though the mass term (30) has a longitudinal part. The diagram (32) can be evaluated for arbitrary dimension D , e.g., by introducing Feynman parameters. We will here concentrate on $D = 2$, which will turn out to play a particularly important role, and comment on other dimensions later. Then, evaluating (32) in $D = 2$ and expanding the result in powers of (p^2/m^2) for the extreme infrared behavior gives

$$-\frac{1}{2} \frac{N g^2}{4\pi} \delta^{ab} \frac{p^2}{m^2} \left[\ln \frac{p^2}{m^2} - 1 - \frac{1}{2} \frac{p^2}{m^2} + \mathcal{O}\left((p^2/m^2)^2\right) \right]. \quad (33)$$

On the other hand, neglecting the term (31) in the action corresponds to the replacement

$$\frac{1}{k^2 + m^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \longrightarrow \frac{1}{m^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (34)$$

for the tree-level gluon propagator. Making this replacement in the loop diagram (32) leads to the result

$$-\frac{1}{2} \frac{N g^2}{4\pi} \delta^{ab} \frac{p^2}{m^2} \left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln \frac{p^2}{\kappa^2} \right) \quad (35)$$

(plus terms of order ϵ) in $D = 2 + \epsilon$ dimensions, with an arbitrary unit of mass κ . The appearance of an ultraviolet divergency in two dimensions is due to the radical change in the ultraviolet behavior of the gluon propagator. To be precise, the symbol g in eq. (35) stands for the bare coupling constant multiplied with a power of κ such that g has the dimension of mass, and it coincides with the coupling constant appearing in eq. (33) only in the limit $D \rightarrow 2$.

The results (33) and (35) are obviously different. However, if we implement an appropriate normalization condition for the term proportional to p^2 in the proper ghost two-point function, then *after renormalization* the two results will only differ by contributions of the order (p^2/m^2) relative to the leading term, since renormalization takes care of the constants multiplying p^2 . Expressions (33) and (35) are then equivalent in the infrared limit, and in this sense the replacement (34) of the tree-level gluon propagator is justified. The argument can easily be extended to any dimension D in the range $2 \leq D \leq 4$.

We can, hence, neglect the term (31) in the action as long as we are only interested in describing the extreme infrared regime of the theory. These modifications, the introduction of a mass term for the gluons and the subsequent omission of the term (31), have profound consequences for the renormalization group analysis of the theory. We begin this analysis by considering the free ($g = 0$) part of the action which is now given by

$$S_{\text{IR}}^0 = \int d^D x \left(\frac{1}{2} A_\mu^a m^2 A_\mu^a + i B^a \partial_\mu A_\mu^a + \partial_\mu \bar{c}^a \partial_\mu c^a \right). \quad (36)$$

Performing a Wilsonian renormalization group transformation on this free field theory (see, e.g., ref. [35]), the only nontrivial step is the rescaling of the space-time coordinates x and the fields. Writing the rescaling of the coordinates as $x \rightarrow x/s$ with a parameter $s > 1$, the action (36) is invariant under the renormalization group transformation if we rescale the gluon field as

$$A_\mu^a(x) \rightarrow s^{D/2} A_\mu^a(sx). \quad (37)$$

This behavior under rescaling defines the canonical dimension of the gluon field which is hence $D/2$ and manifestly differs from its usual canonical dimension $(D-2)/2$. The reason is that the action (36) corresponds to the high-temperature fixed point rather than the usually considered critical Gaussian fixed point [35], as far as the gluon field is concerned. The high-temperature fixed point is naturally related to the model of the stochastic vacuum. The canonical dimension of the ghost fields, on the other hand, is the ordinary $(D-2)/2$.

Reintroducing the coupling terms of Faddeev-Popov theory into the action, the scaling dimensions around the infrared fixed point action (36) turn out to be different for the different couplings. Concretely, both the three-gluon and the four-gluon couplings become perturbatively *irrelevant*, while the ghost-gluon coupling is relevant only for dimensions $D < 2$ (and marginal at $D = 2$). In other words, the upper critical dimension of the theory is two. This can also be concluded from an infrared power counting analysis to all orders of perturbation theory [34].

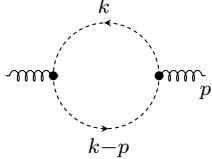
Hence, for an infrared analysis of the theory in dimensions $D > 2$, we would be justified to work with the free action (36) alone, which obviously directly leads to the decoupling solution for the propagators (in the extreme infrared limit). It will turn out to be very interesting, however, to include the coupling that is marginal at $D = 2$ dimensions and to treat dimensions above two in an epsilon expansion. We will hence add the coupling term

$$S_{\text{IR}}^1 = m \kappa^{(2-D)/2} \bar{g} f^{abc} \int d^D x \partial_\mu \bar{c}^a A_\mu^b c^c \quad (38)$$

to the free action (36), where we have introduced the dimensionless coupling constant \bar{g} . Incidentally, keeping the ghost-gluon coupling while neglecting the three- and four-gluon couplings is precisely equivalent to keeping only the diagrams with the biggest number of (internal) ghost propagators, that is, ghost dominance.

5.2. Infrared fixed point

In the theory defined by $S_{\text{IR}}^0 + S_{\text{IR}}^1$, the one-loop expressions for the propagators are precisely those represented in fig. 1, except that the propagators in the loop diagrams should be replaced with the tree-level propagators. We have given the result for the ghost self-energy in $D = 2 + \epsilon$ dimensions already in eq. (35). For the gluon self-energy, we get in $D = 2 + \epsilon$ dimensions



$$= -N m^2 (\kappa^2)^{(2-D)/2} \bar{g}^2 \delta^{ab} \int \frac{d^D k}{(2\pi)^D} (k_\mu - p_\mu) \frac{1}{k^2} k_\nu \frac{1}{(k-p)^2}$$

$$= \frac{1}{2} \frac{N \bar{g}^2}{4\pi} m^2 \delta^{ab} \left[\left(\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) + \ln \frac{p^2}{\kappa^2} - 2 \right) \delta_{\mu\nu} + 2 \frac{p_\mu p_\nu}{p^2} \right]. \quad (39)$$

It is important for the consistency of the renormalization procedure that the counterterm can be chosen local, in this case proportional to $m^2 \delta_{\mu\nu}$. The gluon self-energy is *not* transverse in momentum, which is not a surprise since BRST invariance is broken. Only the transverse part, given by the coefficient of $\delta_{\mu\nu}$ in the expression (39), enters the result for the gluon propagator.

For the renormalization of the theory, we introduce renormalized fields as $A_\mu^a = Z_A^{1/2} A_{R,\mu}^a$ and $c^a = Z_c^{1/2} c_R^a$ and normalize the renormalized propagators to

$$\langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle \Big|_{p^2=\mu^2} = \frac{1}{m^2} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p-q),$$

$$\langle c_R^a(p) \bar{c}_R^b(-q) \rangle \Big|_{p^2=\mu^2} = \frac{1}{\mu^2} \delta^{ab} (2\pi)^D \delta(p-q) \quad (40)$$

at the renormalization scale μ . Using the expressions (35) and (39) for the self-energies, the normalization conditions (40) define the field renormalization constants Z_A and Z_c as functions of the scale μ . The one-loop anomalous dimensions are then easily calculated to be

$$\gamma_A = \mu^2 \frac{d}{d\mu^2} \ln Z_A = \frac{1}{2} \frac{N \bar{g}^2}{4\pi}, \quad \gamma_c = \mu^2 \frac{d}{d\mu^2} \ln Z_c = -\frac{1}{2} \frac{N \bar{g}^2}{4\pi}. \quad (41)$$

We define the dimensionless renormalized coupling constant as usual from the renormalized vertex function at the symmetric point,

$$\Gamma_{R,\bar{c}Ac}(p, q, r) \Big|_{p^2=q^2=r^2=\mu^2} = Z_A^{1/2}(\mu) Z_c(\mu) \Gamma_{\bar{c}Ac}(p, q, r) \Big|_{p^2=q^2=r^2=\mu^2}$$

$$= m \mu^{-\epsilon/2} \bar{g}_R(\mu) f^{abc} i p_\rho (2\pi)^D \delta(p+q+r) \quad (42)$$

(cf. eq. (38)), where we have omitted the Lorentz and color indices on the symbol $\Gamma_{\bar{c}Ac}$ of the vertex function. We can now calculate the beta function. Taylor's non-renormalization theorem implies that the bare vertex function at the symmetric point is independent of the scale μ , so that the one-loop beta function becomes

$$\beta(\epsilon, \bar{g}_R) = \mu^2 \frac{d}{d\mu^2} \bar{g}_R = \frac{1}{2} \bar{g}_R \left(\frac{\epsilon}{2} + \gamma_A + 2\gamma_c \right) = \frac{1}{2} \bar{g}_R \left(\frac{\epsilon}{2} - \frac{1}{2} \frac{N \bar{g}_R^2}{4\pi} \right). \quad (43)$$

From eq. (43) we identify two fixed points for $\epsilon > 0$ ($D > 2$): a trivial infrared-stable one, and an infrared-unstable one at

$$\frac{N \bar{g}_R^2}{4\pi} = \epsilon. \quad (44)$$

We begin with the discussion of the infrared-unstable fixed point. The μ -independence of the bare propagators leads to differential equations for the renormalized propagators: using the definition of the anomalous dimensions we obtain the Callan-Symanzik equations

$$\begin{aligned}\mu^2 \frac{d}{d\mu^2} \langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle &= -\gamma_A \langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle, \\ \mu^2 \frac{d}{d\mu^2} \langle c_R^a(p) \bar{c}_R^b(-q) \rangle &= -\gamma_c \langle c_R^a(p) \bar{c}_R^b(-q) \rangle.\end{aligned}\tag{45}$$

Inserting the fixed point value (44) of the coupling constant in the anomalous dimensions, the differential equations for the renormalized propagators may be integrated with the normalization conditions (40) as initial conditions. The results are

$$\begin{aligned}\langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle &= \frac{1}{m^2} \left(\frac{p^2}{\mu^2} \right)^{\epsilon/2} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p-q), \\ \langle c_R^a(p) \bar{c}_R^b(-q) \rangle &= \frac{1}{p^2} \left(\frac{\mu^2}{p^2} \right)^{\epsilon/2} \delta^{ab} (2\pi)^D \delta(p-q).\end{aligned}\tag{46}$$

This is *exactly* the infrared behavior of the first scaling solution (17). So the renormalization group is able to reproduce one of the scaling solutions in the infrared regime and furthermore demonstrates that it corresponds to an infrared-*unstable* fixed point.

We should clarify that we refer to eqs. (45) as (simple examples of) Callan-Symanzik equations for lack of a better name and following the (rather recent) general use of the term, see, e.g., ref. [35]. However, these are not precisely the equations originally derived by Callan and Symanzik, but rather homogeneous renormalization group equations that result from the reparameterization invariance of the bare correlation functions.

Now we turn to the trivial infrared-stable fixed point. The fact that the stable fixed point is at $\bar{g}_R = 0$ implies that perturbation theory with the action $S_{\text{IR}}^0 + S_{\text{IR}}^1$ should give a good description of the infrared regime. In fact, it was shown in refs. [33, 34] that one-loop perturbation theory reproduces the lattice results very well in four dimensions and at least qualitatively in three dimensions. The one-loop contributions that include the three- and four-gluon vertices were taken into account in this comparison, however, they are suppressed by powers of (p^2/m^2) for small momenta.

Actually, the renormalization group approach can be used to go beyond the mere fact that the decoupling solution in the extreme infrared limit corresponds to a stable fixed point. The way the coupling constant approaches the fixed point value (zero) can be obtained by integrating the differential equation (43) for $\bar{g}_R(\mu)$. The result is

$$\frac{N\bar{g}_R^2(\mu)}{4\pi} = \frac{(\mu^2/\Lambda^2)^{\epsilon/2}}{1 + (\mu^2/\Lambda^2)^{\epsilon/2}} \epsilon,\tag{47}$$

where we have introduced a new characteristic scale Λ . It is defined as

$$\frac{N\bar{g}_R^2(\Lambda)}{4\pi} = \frac{\epsilon}{2}\tag{48}$$

and should eventually be related to Λ_{QCD} (this relation, however, is beyond the scope of the present approach). Substituting the result (47) for \bar{g} in the anomalous dimensions (41), the

integration of the differential equations (45) yields

$$\begin{aligned}\langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle &= \frac{1}{m^2} \frac{1 + (p^2/\Lambda^2)^{\epsilon/2}}{1 + (\mu^2/\Lambda^2)^{\epsilon/2}} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p - q), \\ \langle c_R^a(p) \bar{c}_R^b(-q) \rangle &= \frac{1}{p^2} \frac{1 + (\mu^2/\Lambda^2)^{\epsilon/2}}{1 + (p^2/\Lambda^2)^{\epsilon/2}} \delta^{ab} (2\pi)^D \delta(p - q).\end{aligned}\quad (49)$$

The infrared behavior of the propagators predicted by eq. (49) is at least qualitatively confirmed by lattice calculations in $D = 3$ dimensions [36]. In $D = 4$ space-time dimensions ($\epsilon = 2$), the increase of the gluon propagator proportional to p^2 starting from a non-zero constant at $p^2 = 0$ as described by eq. (49) is actually seen [37] in lattice calculations at lattice parameter $\beta = 0$, which in continuum language means that the proper Yang-Mills action

$$\int d^D x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \quad (50)$$

is omitted in the probability distribution, and only the delta functional (1) and the Faddeev-Popov determinant (2) (and the restriction to the Gribov region) are left.

In order to reproduce the $D = 4$ lattice results with the Yang-Mills action taken into account, we have to reintroduce the term (31) into the action $S_{\text{IR}}^0 + S_{\text{IR}}^1$. The reason is that this term, although perturbatively irrelevant given the scaling behavior (37) of the gluon field, is of the same order in (p^2/m^2) as the correction to the tree-level gluon propagator obtained from the renormalization group improvement in eq. (49) for $\epsilon = 2$. Since at the one-loop level there are no quantum corrections to (31) considered as a composite operator, we can just add this term to the renormalization-group improved result (49) for the proper gluonic two-point function (inverse propagator for the transverse part), which results in

$$\langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle = \left(p^2 + m^2 \frac{1 + \mu^2/\Lambda^2}{1 + p^2/\Lambda^2} \right)^{-1} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p - q). \quad (51)$$

This is precisely of the form (25) of the infrared propagator found in the refined Gribov-Zwanziger framework (without an additional $\langle A_\mu^a A_\mu^a \rangle$ -condensate), with the effective momentum-dependent mass given by (cf. eqs. (27), (28))

$$M_{\text{eff}}^2(p^2) = \frac{m^2(\mu^2 + \Lambda^2)}{p^2 + \Lambda^2}. \quad (52)$$

A different renormalization scheme based on the full Faddeev-Popov action (4) plus the gluon mass term (30) has been proposed in ref. [34]. The corresponding Callan-Symanzik equations reproduce the decoupling solutions as measured on the lattice reasonably well over the whole momentum range in four dimensions, and at least qualitatively in three dimensions.

In the special case of two space-time dimensions ($\epsilon = 0$), the two fixed points considered so far coalesce, and the result is one infrared-unstable trivial fixed point. The instability implies that the free action (36) cannot describe the infrared physics in two dimensions. Indeed, calculations on the lattice confirm that in two dimensions the decoupling solution corresponding to the action (36) is *not* realized [25]. Instead, apparently the second scaling solution (18) is found, and we have not yet succeeded in giving a description of the corresponding fixed point (see, however, below).

5.3. Lifshitz point

Up to now, we have not made any attempt to implement the horizon condition in the form (22). It is not difficult to do so: on the classical level, for a local (and covariant) formulation, we have to replace the ghost term in the action (36) with

$$\int d^D x \frac{1}{b^2} \partial_\mu \bar{c}^a (-\partial^2) \partial_\mu c^a, \quad (53)$$

where we have introduced an arbitrary dimensionful parameter b . The form (53) of the action is known in statistical physics as an isotropic Lifshitz point [38, 39].

Using the term (53) in the free action changes the canonical dimension of the ghost fields. Scale invariance of the free action now implies the scaling behavior

$$c^a(x) \rightarrow s^{(D-4)/2} c^a(sx), \quad \bar{c}^a(x) \rightarrow s^{(D-4)/2} \bar{c}^a(sx). \quad (54)$$

Then, around the Lifshitz point, the three- and four-gluon couplings are perturbatively irrelevant just as before, while the ghost-gluon coupling becomes relevant for dimensions $D < 6$.

In order to describe the infrared behavior of the theory with the horizon condition (22) implemented in dimensions between two and four, we use an epsilon expansion around $D = 6$. We introduce the coupling term

$$S_{\text{IR}}^1 = \frac{m}{b^2} \kappa^{(6-D)/2} \bar{g} f^{abc} \int d^D x \partial_\mu \bar{c}^a A_\mu^b c^c \quad (55)$$

with the dimensionless coupling constant \bar{g} (different from the coupling constant in eq. (38)), and follow the procedure applied before in close analogy, starting with the calculation of the ghost and gluon self-energies, now at dimension $D = 6 - \epsilon$ and with the tree-level ghost propagator proportional to b^2/k^4 . We stress that a fine tuning of the tree-level ghost term that appears in eq. (36) is necessary in order to fulfill the horizon condition (22) to one-loop order, just like in ordinary ϕ^4 -theory the tree-level mass term has to be fine-tuned to reach the critical point.

We use normalization conditions and a definition of the renormalized coupling constant $\bar{g}_R(\mu)$ in strict analogy to eqs. (40) and (42), adapted to the new form (53) of the ghost term and the coupling (55). Taylor's non-renormalization theorem continues to hold in the present situation, and the result for the one-loop beta function is

$$\beta(\epsilon, \bar{g}_R) = \mu^2 \frac{d}{d\mu^2} \bar{g}_R = -\frac{1}{2} \bar{g}_R \left(\frac{\epsilon}{2} - \frac{1}{2} \frac{N \bar{g}_R^2}{(4\pi)^3} \right). \quad (56)$$

It thus turns out that the infrared-stable fixed point for dimensions $D < 6$ ($\epsilon > 0$) is the nontrivial one. The final result for the renormalization-group improved infrared propagators corresponding to the infrared-stable fixed point is

$$\begin{aligned} \langle A_{R,\rho}^a(p) A_{R,\sigma}^b(-q) \rangle &= \frac{1}{m^2} \left(\frac{p^2}{\mu^2} \right)^{\epsilon/12} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} \right) \delta^{ab} (2\pi)^D \delta(p-q), \\ \langle c_R^a(p) \bar{c}_R^b(-q) \rangle &= \frac{b^2}{p^4} \left(\frac{p^2}{\mu^2} \right)^{5\epsilon/24} \delta^{ab} (2\pi)^D \delta(p-q). \end{aligned} \quad (57)$$

In the notation of eq. (16), we obtain the following values for the infrared exponents:

$$\alpha_F(D) = \frac{5D-6}{24}, \quad \alpha_G(D) = -\frac{18-D}{12}. \quad (58)$$

The exponents fulfill the sum rule (19). The values (58) should be compared to the (approximate) exponents of the second scaling solution (18),

$$\alpha_F(D) = \frac{5D - 5}{25}, \quad \alpha_G(D) = -\frac{16 - D}{10}. \quad (59)$$

The two results coincide (exactly) at $D = 6$, and in the range $2 \leq D \leq 4$ they differ the most at $D = 2$ (where the approximation (59) turns out to be exact): $\alpha_F = 1/6$ vs. $\alpha_F = 1/5$ and $\alpha_G = -8/6$ vs. $\alpha_G = -7/5$. However, compared to our experience with the epsilon expansion in statistical physics, the coincidence of the infrared exponents (58) and (59) is unexpectedly good, and we consider them to be approximations to the same solution of the exact Dyson-Schwinger equations of the theory.

5.4. Discussion

We have reproduced the two scaling solutions and the decoupling solution of the Dyson-Schwinger equations with the help of an analytical renormalization group analysis. Only the second scaling solution (18) and the decoupling solution correspond to infrared-stable fixed points, while the first scaling solution (17) corresponds to an infrared-unstable fixed point in dimensions D between two and four.

In fact, the Lifshitz point which leads to the second scaling solution is only stable if one implements the horizon condition (22) (in addition to the gluon mass term which is a natural result of the breaking of BRST invariance). As remarked after eq. (55), the implementation of the horizon condition requires a fine tuning of the tree-level term

$$\int d^D x \partial_\mu \bar{c}^a \partial_\mu c^a, \quad (60)$$

a relevant coupling under the scaling behavior (54) of the ghost fields. Consequently, the nontrivial fixed point in eq. (56) is infrared-*unstable* unless one insists in implementing the horizon condition.

We hence conclude that in dimensions $D > 2$ the physically realized solution according to lattice simulations and the refined Gribov-Zwanziger approach is the only one that corresponds to an infrared-stable fixed point. In other words, the only input in the Callan-Symanzik approach is the gluon mass term that is generated by the breaking of BRST invariance, then the renormalization group flow automatically leads to the decoupling solution without further assumptions. In two space-time dimensions, on the other hand, the trivial fixed point corresponding to the decoupling solution becomes unstable, and the lattice results point to the second scaling solution in this case. However, according to the explanation above the fixed point corresponding to this scaling solution should also be unstable. Indeed, calculating the one-loop corrections to (60) considered as a composite operator, the fixed point becomes even more strongly infrared-repulsive with decreasing dimension. Hence, unless there is some reason to implement the horizon condition precisely in two dimensions, we do not yet have a satisfactory understanding of the infrared behavior of the theory in two dimensions within the Callan-Symanzik approach.

In conclusion, we have presented a new approach to the infrared regime of Yang-Mills theory in Landau gauge that is based on a renormalization group analysis with Callan-Symanzik equations in an epsilon expansion. In contrast to the Dyson-Schwinger equations, the Callan-Symanzik equations are analytical and inherently systematic as an approximation scheme. Furthermore, they are technically relatively simple and economic as compared to the Gribov-Zwanziger approach with the additional auxiliary fields necessary for a local formulation. This should be an important advantage in the further elaboration of the theory, in particular the inclusion of condensates (as in the refined Gribov-Zwanziger framework) and dynamical quarks.

Acknowledgments

Support by CIC-UMSNH and Conacyt project CB-2009/131787 is gratefully acknowledged.

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